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On the general theory of contained rotating fluid motions

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A general linear theory is developed to describe the manner in which rigid fluid rotation is established from a prescribed initial state of motion in a container of arbitrary shape. The container rotates with uniform angular velocity and is filled with a viscous incompressible fluid.

A new mean circulation theorem is proved and used to separate the flow into geostrophic motion and inertial oscillations. The basic eigenvalue problem is studied and important properties concerning spectrum, orthogonality and completeness are deduced. The effect of viscosity on the inviscid modes is calculated in a manner that maintains the solution uniformly valid through the spin-up time. All modes decay in this time scale which characterizes the entire transition to rigid rotation in all configurations.

1. Introduction

This paper considers the transient flow of a viscous fluid filling an arbitrarily shaped container that rotates with constant angular velocity. Of principal interest is the manner in which any initial state of motion is resolved into rigid rotation. A previous paper, Greenspan (1964) (henceforth denoted by I), dealt with motion inside a sphere because this special, but typical, case allows the complete explicit determination of eigenmodes, frequencies, viscous effects, etc., and is a convenient configuration for experimental purposes. Furthermore, the specific results provide the insight and motivation to formulate the appropriate generalizations and always serve as a concrete illustration of the complete theory. Moreover, the analytical methods used here are extensions of those already developed and employed and in the interests of brevity frequent reference to I is made for pertinent details of extended procedure. However, many of the general theorems and results presented here are new; others are deduced in different and simpler ways than were their counterparts in I. The mean circulation theorem of § 4 exemplifies these remarks.

The basic procedure is almost the same as that used previously and, briefly, is as follows: An expansion in inverse half powers of the Taylor number, R, is introduced in the governing equations and the eigenmodes of the inviscid problem, $R = \infty$, and their properties are determined first. Each mode is then corrected for viscous effects (boundary layers) in a manner designed to achieve a uniformly valid approximation through the spin-up time, $T = L(\Omega\nu)^{-\frac{1}{2}}$ (symbols are defined in § 2). The synthesis of the initial flow into the viscosity-corrected uniformly valid modes completes the solution. An extensive discussion of both method and solution for the special case of the spherical container appears in I; a description of the general case is given in § 5.

2. Formulation

Consider a closed container of arbitrary shape that rotates with uniform angular velocity, Ω, about a fixed axis along which the z co-ordinate is measured. A viscous incompressible fluid fills the container and is, at some initial instant, in a prescribed state of motion that deviates slightly from rigid rotation. If L, $\epsilon\Omega L$ and Ω^{-1} characterize the length (container height), initial velocity, and time respectively, then these quantities may be used to make the equations of motion dimensionless

$$\frac{\partial \mathbf{q}}{\partial t} + \epsilon \mathbf{q} . \nabla \mathbf{q} + 2\mathbf{k} \times \mathbf{q} + \nabla p = R^{-1} \Delta \mathbf{q}, \quad \nabla . \mathbf{q} = 0, \tag{2.1}$$

where $R = \Omega L^2 \nu^{-1} \gg 1$ is the Taylor number and p is the actual pressure less the centrifugal pressure $\frac{1}{2}\rho\Omega L^2(\mathbf{r} - \mathbf{k}z)^2$. The Rossby number, ϵ, is assumed to be small. (In fact, non-linear effects have been shown to be relatively unimportant in one case compared to the action of viscosity for rather significant variation of the parameter ϵ, Greenspan & Weinbaum 1965.) Accordingly, only the linear problem is considered and the fundamental boundary-value problem becomes

$$\frac{\partial \mathbf{q}}{\partial t} + 2\mathbf{k} \times \mathbf{q} + \nabla p = R^{-1} \Delta \mathbf{q}, \quad \nabla . \mathbf{q} = 0, \tag{2.2}$$

with $\mathbf{q} = 0$ at the boundary and $\mathbf{q}(\mathbf{r}, 0) = \mathbf{q}_*(\mathbf{r})$ is the initial velocity distribution. Of course, $\mathbf{q}_*$ satisfies the mass conservation law and the boundary condition $\mathbf{n} . \mathbf{q}_* = 0$ at the container wall ($\mathbf{n}$ is the unit outward normal to the bounding surface).

A single equation for the pressure is obtainable from (2.2)

$$\left(\frac{\partial}{\partial t} - R^{-1}\nabla^2\right)^2 \nabla^2 p + 4 \frac{\partial^2 p}{\partial z^2} = 0, \tag{2.3}$$

but the boundary conditions cannot be written in terms of just one dependent scalar variable.

It is known that, for R very large, direct viscous action is confined to a thin boundary layer at the container wall of thickness $R^{-\frac{1}{2}}$. However, this Ekman layer produces a small $O(R^{-\frac{1}{2}})$ circulation in the inviscid interior which exerts a major influence on the flow by means of vortex line stretching and the transport of angular momentum. In this fashion, viscosity eliminates the initial velocity distribution in the spin-up time scale, $R^{\frac{1}{2}}$. Thus, if all the important phenomena are to be included in the analysis, the solution (accurate to $O(R^{-\frac{1}{2}})$) of the fundamental boundary-value problem, (2.2) *et seq.*, must be uniformly valid in space and time at least through spin-up.

The third time scale, $t = O(R)$, characterizing the time required for viscous diffusion to permeate into the interior, is of little importance in transient motions.

However, non-uniformities in the boundary layers themselves are important but fortunately they produce lower-order effects than concern us here. These effects do make obscure the correct form of the expansions appearing below, beyond terms of order $R^{-\frac{1}{2}}$, and a complete formal expansion procedure is not attempted.

Let $\mathbf{n}$ be the unit outwardly directed normal at the bounding wall and let ζ be the stretched boundary-layer co-ordinate so that in the viscous layer

$$\mathbf{n}.\nabla = -R^{\frac{1}{2}}(\partial/\partial\zeta)$$

but $\mathbf{n}\times\nabla$ is an order-one tangential derivative. The container itself is described as consisting of top and bottom surfaces given by

$$\left.\begin{aligned} z &= f(x,y) \quad \text{(top)},\\ z &= -g(x,y) \quad \text{(bottom)}.\end{aligned}\right\} \tag{2.4}$$

A solution is sought in the form of a superposition of all the natural modes of vibration of the inviscid problem (equation (2.2) with $R=\infty$), each of which is corrected for the effects of viscosity in a manner consistent with the requirement of uniform validity through spin-up. Therefore, let

$$\left.\begin{aligned} \mathbf{q} &= \mathbf{q}_0(\mathbf{r},R^{-\frac{1}{2}}t)+\sum_m \mathbf{Q}_m(\mathbf{r})\,e^{s_m t}+R^{-\frac{1}{2}}\{\mathbf{q}_1(\mathbf{r},R^{-\frac{1}{2}}t)+\sum_m \mathbf{q}_{m1}(\mathbf{r},t)\}+\dots\\ &\qquad +\tilde{\mathbf{q}}_0+\sum_m \tilde{\mathbf{q}}_m+R^{-\frac{1}{2}}\{\tilde{\mathbf{q}}_1+\sum_m \tilde{\mathbf{q}}_{m1}\}+\dots,\\ p &= \phi_0(\mathbf{r},R^{-\frac{1}{2}}t)+\sum_m \Phi_m(\mathbf{r})\,e^{s_m t}+R^{-\frac{1}{2}}\{\phi_1(\mathbf{r},R^{-\frac{1}{2}}t)+\sum_m \phi_{m1}(\mathbf{r},t)\}+\dots\\ &\qquad +\tilde{\phi}_0+\sum_m \tilde{\phi}_m+R^{-\frac{1}{2}}\{\tilde{\phi}_1+\sum_m \tilde{\phi}_{m1}\}+\dots,\end{aligned}\right\} \tag{2.5}$$

where the tilde symbol denotes a boundary-layer function (a function of ζ that approaches zero exponentially as $\zeta\to\infty$). To achieve uniform validity for t large, the parameter s_m must also be expanded as

$$s_m = s_{m0}+R^{-\frac{1}{2}}s_{m1}+\dots, \tag{2.6}$$

where $s_{m0}=i\lambda_m$ represents an inviscid eigenvalue and s_{m1} is to be chosen to eliminate secular terms possessing unacceptable growth rates. The procedure is very similar, but not identical, to the classical method of Poincaré. The spin-up mode $\mathbf{q}_0(\mathbf{r},R^{-\frac{1}{2}}t)$, which in a sense corresponds to zero eigenvalues, requires special treatment. The flow represented by $\mathbf{q}_0$, ϕ_0 is more appropriately referred to as a geostrophic motion because it results from a balance of pressure and Coriolis forces.

The substitution of these expansions into the basic equations and requisite boundary conditions leads to a sequence of problems for the inviscid and boundary-layer flows and their mutual interactions. For the spin-up mode, this sequence is the following:

A_1: $\quad \partial(\tilde{\mathbf{q}}_0.\mathbf{n})/\partial\zeta = 0, \quad (\tilde{\mathbf{q}}_0.\mathbf{n}=0); \quad \tilde{\phi}_0 = 0;$

A_2: $\quad 2\mathbf{k}\times\mathbf{q}_0 = -\nabla\phi_0, \quad \nabla.\mathbf{q}_0 = 0,$

with $\mathbf{q}_0 \cdot \mathbf{n} = 0$ on the boundary;

$$A_3\colon\quad 2\mathbf{k}\times\tilde{\mathbf{q}}_0 - \mathbf{n}\,\partial\tilde{\phi}_1/\partial\zeta = \partial^2\tilde{\mathbf{q}}_0/\partial\zeta^2; \quad -\partial(\mathbf{n}\cdot\tilde{\mathbf{q}}_1)/\partial\zeta + \mathbf{n}\cdot\nabla\times(\mathbf{n}\times\tilde{\mathbf{q}}_0) = 0;$$

with $\tilde{\mathbf{q}}_0 = -\mathbf{q}_0$ on the boundary $\zeta = 0$;

$$A_4\colon\quad \partial\mathbf{q}_0/\partial\tau + 2\mathbf{k}\times\mathbf{q}_1 = -\nabla\phi_1 \quad (\tau = R^{-\frac{1}{2}}t); \quad \nabla\cdot\mathbf{q}_1 = 0,$$

with $\mathbf{n}\cdot(\tilde{\mathbf{q}}_1 + \mathbf{q}_1) = 0$ on the boundary and $\mathbf{q}_0(\mathbf{r}, 0)$ prescribed.

Physically speaking, the basic interior flow (problem A_2) requires a boundary layer (A_3) to reduce the tangential velocity to zero at the wall. In turn, this boundary layer produces an $O(R^{-\frac{1}{2}})$ normal flux which forces further interior motion of the same scale (A_4). The tangential velocity of this secondary circulation must be corrected at the boundary and so on and so forth.

The problem sequence for the typical inertial mode represented as

$$\mathbf{q} = \mathbf{Q}_m e^{s_m t} + R^{-\frac{1}{2}}\mathbf{q}_{m1} + \ldots + \tilde{\mathbf{q}}_m + R^{-\frac{1}{2}}\tilde{\mathbf{q}}_{m1} + \ldots,$$

$$p = \Phi_m e^{s_m t} + R^{-\frac{1}{2}}\phi_{m1} + \ldots + \tilde{\phi}_m + R^{-\frac{1}{2}}\tilde{\phi}_{m1} + \ldots,$$

$$s_m = i\lambda_m + s_{m1}R^{-\frac{1}{2}} + \ldots,$$

is the following:

$$B_1\colon\quad \partial(\tilde{\mathbf{q}}_m\cdot\mathbf{n})/\partial\zeta = 0, \quad \tilde{\phi}_m = 0;$$

$$B_2\colon\quad i\lambda_m\mathbf{Q}_m + 2\mathbf{k}\times\mathbf{Q}_m + \nabla\Phi_m = 0, \quad \nabla\cdot\mathbf{Q}_m = 0,$$

with $\mathbf{Q}_m\cdot\mathbf{n} = 0$ on the boundary;

$$B_3\colon\quad \partial\,\tilde{\mathbf{q}}_m/\partial t + 2\mathbf{k}\times\tilde{\mathbf{q}}_m - \mathbf{n}\,\partial\tilde{\phi}_1/\partial\zeta = \partial^2\tilde{\mathbf{q}}_m/\partial\zeta^2,$$

$$-\partial(\mathbf{n}\cdot\tilde{\mathbf{q}}_{m1})/\partial\zeta + \mathbf{n}\cdot\nabla\times(\mathbf{n}\times\tilde{\mathbf{q}}_m) = 0,$$

with $\tilde{\mathbf{q}}_m = -\mathbf{Q}_m e^{s_m t}$ on $\zeta = 0$ and $(\mathbf{q}_m)_{t=0} = 0$;

$$B_4\colon\quad \partial\mathbf{q}_{m1}/\partial t + 2\mathbf{k}\times\mathbf{q}_{m1} = -\nabla\phi_{m1} - s_{m1}\mathbf{Q}_m e^{s_m t}, \quad \nabla\cdot\mathbf{q}_{m1} = 0,$$

with $\mathbf{n}\cdot(\mathbf{q}_{m1} + \tilde{\mathbf{q}}_{m1}) = 0$ on the boundary; $(\mathbf{q}_{m1})_{t=0} = 0$.

It would, of course, be simplest to assume the same exponential time behaviour for all functions appearing in problem sequence (B) but this classical procedure encounters certain mathematical difficulties that can be surmounted, but in a none too satisfactory manner. These difficulties arise from interchange of limit processes and integrations involving R and ζ at particular critical positions on the container surface given by $2\mathbf{n}\cdot\mathbf{k} = \lambda$ (critical latitudes in the sphere problem). The present method avoids this trouble in a more satisfactory manner, produces a superior approximation for the actual time-dependent boundary layers, and leads to the same direct calculation for the decay factor s_{m1} as does the classical analysis. However, if only the parameter s_{m1} is desired, we may use the classical analysis and dismiss all the difficulties that arise (divergent integrals at one stage) by appealing to the more elaborate procedure, initiated here and developed in I, for some justification of method.

3. The geostrophic mode

Problem A_2 for the functions $\mathbf{q}_0$ and ϕ_0

$$2\mathbf{k}\times\mathbf{q}_0 = -\nabla\phi_0, \quad \nabla.\mathbf{q}_0 = 0, \tag{3.1}$$

with

$$\mathbf{q}_0.\mathbf{n} = 0 \quad \text{on boundary,}$$

is a special case of the general mode problem B_2 corresponding to $\lambda = 0$. This implies that these functions have no $O(1)$ time dependence and accounts for the assumed form of solution involving the longer time scale $\tau = R^{-\frac{1}{2}}t$ appropriate for spin-up (Greenspan & Howard 1963).

The curl of (3.1) yields

$$(\mathbf{k}.\nabla)\,\mathbf{q}_0 = 0 \quad (\mathbf{q}_0 = \mathbf{q}_0(x,y,\tau)), \tag{3.2}$$

showing that $\mathbf{q}_0$ is a three-dimensional vector independent of height, z. The motion is columnar; the entire column of fluid from the lower surface $z = -g(x,y)$ to the upper surface $z = f(x,y)$ moves as a unit. Furthermore, if the horizontal velocity is introduced

$$\mathbf{q}_0 = \mathbf{q}_0^H + w\mathbf{k}$$

then, upon multiplying (3.1) vectorially by $\mathbf{k}$, it follows that

$$\mathbf{q}_0^H = \tfrac{1}{2}\mathbf{k}\times\nabla\phi_0. \tag{3.3}$$

The axial velocity component is determined from the boundary condition $\mathbf{q}_0.\mathbf{n} = 0$ at both ends of the column. Thus with

$$\mathbf{n}_T = (\mathbf{k}-\nabla f)\{1+(\nabla f)^2\}^{-\frac{1}{2}}; \quad \mathbf{n}_B = (\mathbf{k}+\nabla g)\{1+(\nabla g)^2\}^{\frac{1}{2}}, \tag{3.4}$$

we find that

$$w = \mathbf{q}_0^H.\nabla f = -\mathbf{q}_0^H.\nabla g,$$

or

$$w = \tfrac{1}{2}\mathbf{q}_0^H.(\nabla f - \nabla g). \tag{3.5}$$

Moreover, by subtracting the two equations and using (3.3), it is established that

$$\mathbf{k}.\nabla\phi_0\times\nabla(f+g) = 0.$$

However, ϕ_0 is independent of z, and this result implies that the pressure is a function only of total height, with time acting as a parameter

$$\phi_0 = \phi_0(f+g,\tau) = \phi_0(h,\tau). \tag{3.6}$$

The velocity is then calculated to be

$$\mathbf{q}_0 = -\tfrac{1}{2}[\partial\phi_0(h,\tau)/\partial h]\{1+(\nabla f)^2\}^{\frac{1}{2}}\{1+(\nabla g)^2\}^{\frac{1}{2}}(\mathbf{n}_T\times\mathbf{n}_B). \tag{3.7}$$

Thus, in geostrophic flow a column of fluid of height h moves about the interior of the container as a unit, preserving its length. This mode can exist only if the container has closed contours of constant total height. The simple container consisting of a hemisphere rotating about a diameter on its flat surface is one for which no spin-up mode or geostrophic flow is possible. Constant-depth contours in this case do not form closed streamlines.

The line contour C traced out on the bounding surface (either top or bottom) by a column of constant height, as it moves about, plays a crucial role in the theory to follow. One of the properties of this curve to bear in mind is that its tangent vector is

$$\boldsymbol{\nu} = \mathbf{n}_T \times \mathbf{n}_B,$$

which is of course also the direction of the velocity vector $\mathbf{q}_0$ in space. It is worth restating that $\mathbf{q}_0$ is a three-component vector independent of the z co-ordinate; C is a space contour. For future use, the circulation Γ about C is now recorded,

$$\left.\begin{aligned} \Gamma = \Gamma(h,\tau) &= \oint_C \mathbf{q}_0 . d\mathbf{s}, \\ &= -\frac{1}{2}\frac{\partial}{\partial h}\phi_0(h,\tau)\oint_C ds\{1+(\nabla f)^2\}^{\frac{1}{2}}\{1+(\nabla g)^2\}^{\frac{1}{2}}. \end{aligned}\right\} \tag{3.8}$$

Problems A_3 and A_4 must be solved in order to study the viscous effects on this basic geostrophic flow. Consider the former first:

$$2\mathbf{k}\times\tilde{\mathbf{q}}_0 - \mathbf{n}\,(\partial\tilde{\phi}_1/\partial\zeta) = \partial^2\tilde{\mathbf{q}}_0/\partial\zeta^2, \tag{3.9}$$

$$-\partial\,(\mathbf{n}.\tilde{\mathbf{q}}_1)/\partial\zeta + \mathbf{n}.\nabla\times(\mathbf{n}\times\tilde{\mathbf{q}}_0) = 0, \tag{3.10}$$

with

$$\tilde{\mathbf{q}}_0 + \mathbf{q}_0 = 0 \quad \text{on} \quad \zeta = 0.$$

This is the usual formulation for the Ekman boundary layer. The first equation arises from the requirement that the tangential component of the interior velocity be reduced to zero at the wall. The second equation determines the normal flux, $\mathbf{n}.\mathbf{q}_1$, induced by this viscous layer which produces further interior motion (problem A_4).

Simple vector manipulations of (3.9) lead to the basic boundary-layer equation

$$\partial^2(\mathbf{n}\times\tilde{\mathbf{q}}_0 + i\tilde{\mathbf{q}}_0)/\partial\zeta^2 = 2i(\mathbf{n}.\mathbf{k})\,(\mathbf{n}\times\tilde{\mathbf{q}}_0 + i\tilde{\mathbf{q}}_0), \tag{3.11}$$

whose solution is

$$\mathbf{n}\times\tilde{\mathbf{q}}_0 + i\tilde{\mathbf{q}}_0 = -(\mathbf{n}\times\mathbf{q}_0 + i\mathbf{q}_0)\exp\left[-(2i\mathbf{n}.\mathbf{k})^{\frac{1}{2}}\zeta\right], \tag{3.12}$$

where the positive root is implied. The normal flux into the boundary layer is found by integrating (3.10),

$$(\mathbf{n}.\tilde{\mathbf{q}}_1)_{\zeta=0} = \tfrac{1}{2}\mathbf{n}.\nabla\times\{[\mathbf{n}\times\mathbf{q}_0 + (\mathbf{n}.\mathbf{k})\,\mathbf{q}_0/|\mathbf{n}.\mathbf{k}|]\,|\mathbf{n}.\mathbf{k}|^{-\frac{1}{2}}\}. \tag{3.13}$$

Problem A_4 may now be stated in its entirety

$$2\mathbf{k}\times\mathbf{q}_1 = -\nabla\phi_1 - \partial\mathbf{q}_0/\partial\tau; \tag{3.14}$$

$$\nabla.\mathbf{q}_1 = 0; \tag{3.15}$$

with $\mathbf{n}.\mathbf{q}_1 = (-\mathbf{n}.\tilde{\mathbf{q}}_1)_{\zeta=0}$ at the boundary. The curl of (3.14) reduces it to

$$\partial\mathbf{q}_1/\partial z = \tfrac{1}{2}\partial(\nabla\times\mathbf{q}_0)/\partial\tau, \tag{3.16}$$

or

$$\mathbf{q}_1 = \tfrac{1}{2}z\,\partial(\nabla\times\mathbf{q}_0)/\partial\tau + \mathbf{A}(x,y,\tau).$$

In order that (3.15) be satisfied

$$\mathbf{A} = \tfrac{1}{2}\mathbf{k}\times\partial\mathbf{q}_0/\partial\tau + \nabla\times\mathbf{B}(x,y,\tau),$$

so that

$$\mathbf{q}_1 = \nabla\times(\tfrac{1}{2}z\,\partial\mathbf{q}_0/\partial\tau) + \nabla\times\mathbf{B}. \tag{3.17}$$

A single equation relating $\mathbf{q}_0$ and its time derivative is obtained by applying Stokes's theorem to the top and bottom boundary surfaces ($z = f$, $z = -g$)

enclosed by the spin-up contour C. If S_T, S_B denote the top and bottom surfaces, respectively, $d\mathbf{s} = \boldsymbol{\nu}\, ds = (\mathbf{n}_T \times \mathbf{n}_B)\, ds$ is the directed arc length along C, and all vectors are independent of z, then from (3.17) (properly accounting for sign)

$$\left.\begin{aligned}\int \mathbf{n}_T . \mathbf{q}_1\, dS_T &= \frac{1}{2}\oint_C f \frac{\partial}{\partial \tau}\mathbf{q}_0 . d\mathbf{s} + \oint_C \mathbf{B} . d\mathbf{s},\\ \int \mathbf{n}_B . \mathbf{q}_1\, dS_B &= \frac{1}{2}\oint_C g \frac{\partial}{\partial \tau}\mathbf{q}_0 . d\mathbf{s} - \oint_C \mathbf{B} . d\mathbf{s}.\end{aligned}\right\} \tag{3.18}$$

However, (3.13) and (3.15) imply that

$$\left.\begin{aligned}\int dS_T\, \mathbf{n}_T . \mathbf{q}_1 &= -\frac{1}{2}\oint_C d\mathbf{s} . (\mathbf{n}_T \times \mathbf{q}_0 + \mathbf{q}_0)\, |\mathbf{n}_T . \mathbf{k}|^{-\frac{1}{2}},\\ \int dS_B\, \mathbf{n}_B . \mathbf{q}_1 &= \frac{1}{2}\oint_C d\mathbf{s} . (\mathbf{n}_B \times \mathbf{q}_0 - \mathbf{q}_0)\, |\mathbf{n}_B . \mathbf{k}|^{-\frac{1}{2}}.\end{aligned}\right\} \tag{3.19}$$

The replacement of these expressions in the preceding set, and the elimination of the integral involving $\mathbf{B}$, yield

$$\begin{aligned}-\oint_C d\mathbf{s} . [(\mathbf{q}_0 + \mathbf{n}_T \times \mathbf{q}_0)\, |\mathbf{n}_T . \mathbf{k}|^{-\frac{1}{2}} + (\mathbf{q}_0 - \mathbf{n}_B \times \mathbf{q}_0)\, |\mathbf{n}_B . \mathbf{k}|^{-\frac{1}{2}}]\\ = \oint_C (f+g) \frac{\partial}{\partial \tau}\mathbf{q}_0 . d\mathbf{s}.\end{aligned} \tag{3.20}$$

Since $d\mathbf{s} \times \mathbf{q}_0 = 0$, and $f+g = h$, a constant on contour C, (3.20) simplifies to

$$-\oint_C d\mathbf{s} . \mathbf{q}_0 (|\mathbf{n}_T . \mathbf{k}|^{-\frac{1}{2}} + |\mathbf{n}_B . \mathbf{k}|^{-\frac{1}{2}}) = h \oint \frac{\partial \mathbf{q}_0}{\partial \tau} . d\mathbf{s}. \tag{3.21}$$

Finally, using equation (3.7) for the geostrophic velocity $\mathbf{q}_0$, this becomes

$$\begin{aligned}-\frac{\partial}{\partial h}\phi_0(h,\tau) \oint_C ds (|\mathbf{n}_T . \mathbf{k}|^{-\frac{1}{2}} + |\mathbf{n}_B . \mathbf{k}|^{-\frac{1}{2}})\, (1 + (\nabla f)^2)^{\frac{1}{2}}\, (1 + (\nabla g)^2)^{\frac{1}{2}}\\ = h \frac{\partial^2 \phi_0}{\partial h\, \partial \tau}(h,\tau) \oint_C (1 + (\nabla f)^2)^{\frac{1}{2}}\, (1 + (\nabla g)^2)^{\frac{1}{2}}\, ds,\end{aligned} \tag{3.22}$$

from which $\partial \phi_0 / \partial h$ can be determined by integration. Therefore

$$\frac{\partial \phi_0(h,\tau)}{\partial h} = \frac{\partial \phi_0(h,0)}{\partial h} \exp\left(-\frac{I}{J}\tau\right), \tag{3.23}$$

where
$$\left.\begin{aligned}I &= \oint_C ds \{1 + (\nabla f)^2\}^{\frac{1}{2}} \{1 + (\nabla g)^2\}^{\frac{1}{2}} [\{1 + (\nabla f)^2\}^{\frac{1}{4}} + \{1 + (\nabla g)^2\}^{\frac{1}{4}}],\\ J &= h \oint_C ds \{1 + (\nabla f)^2\}^{\frac{1}{2}} \{1 + (\nabla g)^2\}^{\frac{1}{2}},\end{aligned}\right\} \tag{3.24}$$

and $\tau = R^{-\frac{1}{2}}t$. The geostrophic velocity is obtained from (3.7). Equation (3.23) represents the complete generalization of the analysis of Greenspan & Howard (1963), who considered only axially symmetric containers. (Calling this flow a 'spin-up' mode in the general case is a misnomer for it is no longer the only motion produced when the angular velocity of the container is suddenly changed, as it is for axially symmetric configurations. The 'geostrophic' flow is a more precise title.)

The function $\phi_0(h, 0)$ is arbitrary and must be determined from the initial conditions but before this can be done the inertial modes must be studied.

4. Inertial modes

Consider now problem B_2 for the possible inviscid inertial oscillations of the rotating system

$$i\lambda \mathbf{Q} + 2\mathbf{k} \times \mathbf{Q} + \nabla\Phi = 0, \quad \nabla . \mathbf{Q} = 0, \tag{4.1}$$

with $\mathbf{Q}.\mathbf{n} = 0$ on the boundary. (The subscript notation is omitted for the time being.) The complex velocity vector is expressible in terms of the pressure as

$$(1 - \tfrac{1}{4}\lambda^2)\,\mathbf{Q} = \tfrac{1}{2}\mathbf{k} \times \nabla\Phi - \tfrac{1}{4}i\lambda\nabla\Phi - (\mathbf{k}/i\lambda)\,(\mathbf{k}.\nabla\Phi) \tag{4.2}$$

if $\lambda \neq 0, \pm 2$. The first exception, $\lambda = 0$, corresponds to the geostrophic mode discussed in the last section. The second exception will be considered after development of the general theory.

Poincaré's problem for the pressure alone

$$\nabla^2\Phi - (4/\lambda^2)\,(\mathbf{k}.\nabla)^2\Phi = 0, \tag{4.3}$$

with
$$-\lambda^2\mathbf{n}.\nabla\Phi + 4(\mathbf{n}.\mathbf{k})\,(\mathbf{k}.\nabla\Phi) + 2i\lambda(\mathbf{k}\times\mathbf{n}).\nabla\Phi = 0, \tag{4.4}$$

on the boundary, formed the basis of study in I. Although the explicit determination of modes and frequencies for a particular configuration (e.g. the sphere) necessitates solving for Φ, the theoretical properties of the problem are often more clearly discerned and proved by retaining the velocity vector intact and using the original formulation, (4.1) *et seq.* In this case, for example, no exception for $|\lambda| = 2$ need be made.

Several properties of this eigenvalue problem are now proved. All are generalizations of results contained in I but the derivations for the most part are simpler. In what follows, the complex conjugate of a function ψ is denoted by ψ^+.

Property I. The eigenvalues λ are real and $|\lambda| \leqslant 2$

Note first, in particular, that if $(\mathbf{Q}, \lambda)$ is an eigenfunction–eigenvalue pair so is $(\mathbf{Q}^+, -\lambda)$. Multiply (4.1) by $\mathbf{Q}^+$. and integrate over the volume of the container so that

$$i\lambda\int \mathbf{Q}.\mathbf{Q}^+ dV + 2\int \mathbf{Q}^+.\mathbf{k}\times\mathbf{Q}\,dV = -\int \mathbf{Q}^+.\nabla\Phi\,dV.$$

Since $\mathbf{Q}^+$ satisfies the divergence equation and $\mathbf{Q}^+.\mathbf{n} = 0$ on the boundary

$$\int \mathbf{Q}^+.\nabla\Phi\,dV = \int \mathbf{Q}^+.\mathbf{n}\Phi\,dS - \int \Phi\nabla.\mathbf{Q}^+ dV = 0.$$

Therefore
$$\lambda = 2i\frac{\int \mathbf{Q}^+.(\mathbf{k}\times\mathbf{Q})\,dV}{\int \mathbf{Q}.\mathbf{Q}^+ dV} = -2\frac{\int \mathrm{Im}\,[\mathbf{k}.\mathbf{Q}\times\mathbf{Q}^+]\,dV}{\int \mathbf{Q}.\mathbf{Q}^+ dV}, \tag{4.5}$$

proving the first part of theorem.

Let
$$\mathbf{Q} = \mathbf{A} + \mathbf{B}i.$$

A simple bound for λ is then

$$|\lambda| \leqslant 4\int |\mathbf{k}.\mathbf{A}\times\mathbf{B}|\,dV \Big/ \int (|\mathbf{A}|^2 + |\mathbf{B}|^2)\,dV. \tag{4.6}$$

However
$$2\,|\mathbf{k}.\mathbf{A}\times\mathbf{B}| \leqslant 2\,|\mathbf{A}|\,|\mathbf{B}| \leqslant |\mathbf{A}|^2 + |\mathbf{B}|^2;$$

thus
$$2\int |\mathbf{k}.\mathbf{A}\times\mathbf{B}|\,dV \leqslant \int (|\mathbf{A}|^2 + |\mathbf{B}|^2)\,dV$$

which shows that
$$|\lambda| \leqslant 2.$$

Property II. Orthogonality

Let $(\mathbf{Q}_n, \lambda_n)$, $(\mathbf{Q}_m, \lambda_m)$ be any two eigenfunction–eigenvalue pairs satisfying (4.1) for which $\lambda_n \neq \lambda_m$. From the basic equations, the following expressions can be arranged

$$\left.\begin{aligned} i\lambda_n \mathbf{Q}_m^+ \cdot \mathbf{Q}_n + 2\mathbf{Q}_m^+ \cdot (\mathbf{k} \times \mathbf{Q}_n) &= -\mathbf{Q}_m^+ \cdot \nabla\Phi_n, \\ -i\lambda_m \mathbf{Q}_n \cdot \mathbf{Q}_m^+ + 2\mathbf{Q}_n \cdot (\mathbf{k} \times \mathbf{Q}_m^+) &= -\mathbf{Q}_n \cdot \nabla\Phi_m^+. \end{aligned}\right\} \tag{4.7}$$

If these are added and integrated over the volume V, then with recognition of the fact that $\mathbf{Q}_m^+ \cdot (\mathbf{k} \times \mathbf{Q}_n) = -\mathbf{Q}_n \cdot (\mathbf{k} \times \mathbf{Q}_m^+)$ it follows that

$$(\lambda_n - \lambda_m) \int \mathbf{Q}_m^+ \cdot \mathbf{Q}_n \, dV = 0. \tag{4.8}$$

Since $\lambda_n \neq \lambda_m$, the basic orthogonality property is

$$\int \mathbf{Q}_m^+ \cdot \mathbf{Q}_n \, dV = 0. \tag{4.9}$$

(If $\lambda_n = -\lambda_m$, then $Q_m^+ = Q_n$, a fact already noted.) In terms of the pressure alone, the orthogonality relationship can be written as

$$\int dV \, [\nabla\Phi_n \cdot \nabla\Phi_m^+ + (4/\lambda_n \lambda_m)(\mathbf{k} \cdot \nabla\Phi_n)(\mathbf{k} \cdot \nabla\Phi_m^+)] = 0, \tag{4.10}$$

a special case of which appeared in I. Equation (4.10) is simpler to use in any specific computation because the solution procedure involves the determination of the pressure Φ. On the other hand, (4.9) is more desirable from the theoretical viewpoint.

Property III. Partial expansion theorem

If a velocity distribution consists of only a superposition of inertial modes

$$\mathbf{U} = \Sigma A_m \mathbf{Q}_m, \tag{4.11}$$

(where the pressure is obtained by integrating the basic momentum equation), then it is a direct result of property II that the Fourier coefficients are

$$A_m = \int \mathbf{U} \cdot \mathbf{Q}_m^+ \, dV \Big/ \int \mathbf{Q}_m^+ \cdot \mathbf{Q}_m \, dV. \tag{4.12}$$

In general, an arbitrary velocity distribution must involve geostrophic motion as well as inertial oscillations. The geostrophic mode, which corresponds to the totality of eigenfunctions having zero eigenvalues, is also orthogonal to any inertial mode in the sense of property II but this is not sufficient. There must be some other distinguishing property that differentiates geostrophy from inertial oscillations if the complete synthesis of any initial distribution is to be accomplished. In other words, how is the arbitrary function $\phi_0(h)$ in equations (3.7) and (3.23) to be determined? The answer is contained in the mean circulation theorem.

Mean circulation theorem

Define the average velocity vector

$$\langle \mathbf{Q} \rangle = \int_{-g}^{f} \mathbf{Q} \, dz \tag{4.13}$$

which is a three-component vector independent of the z co-ordinate. Equations (4.1) can each be rewritten in terms of this depth-averaged velocity; for example

$$\int_{-g}^{f} \nabla . \mathbf{Q}\, dz = \nabla . \langle \mathbf{Q} \rangle - \mathbf{Q}_T . \nabla f - \mathbf{Q}_B . \nabla g + (w_T - w_B), \tag{4.14}$$

where subscripts T or B indicate that the attached function is evaluated at the top or bottom surface of the container, and

$$w = \mathbf{k} . \mathbf{Q}.$$

The boundary condition is

$$\left.\begin{aligned} \mathbf{Q} . \mathbf{n} = 0 &= \mathbf{Q}_T . \nabla f - w_T, \\ = 0 &= \mathbf{Q}_B . \nabla g + w_B, \end{aligned}\right\} \tag{4.15}$$

so that the preceding equation may be simplified to

$$\nabla . \langle \mathbf{Q} \rangle = 0. \tag{4.16}$$

The averaging of the momentum equations takes the form

$$i\lambda \langle \mathbf{Q} \rangle + 2\mathbf{k} \times \langle \mathbf{Q} \rangle = -\nabla \int_{-g}^{f} \phi\, dz + \phi_T \mathbf{n}_T + \phi_B \mathbf{n}_B, \tag{4.17}$$

and in particular

$$i\lambda \nabla \times \langle \mathbf{Q} \rangle = \nabla \times \phi_T \mathbf{n}_T + \nabla \times \phi_B \mathbf{n}_B. \tag{4.18}$$

Application of the Stokes theorem to any section of boundary surface S bounded by a contour γ implies that

$$i\lambda \oint_\gamma d\mathbf{s} . \langle \mathbf{Q} \rangle = \oint_\gamma \phi_T \mathbf{n}_T . d\mathbf{s} + \oint_\gamma \phi_B \mathbf{n}_B . d\mathbf{s}.$$

Let γ be the geostrophic contour C of § 3, the surface contour corresponding to constant total height $f+g = h$ along which $d\mathbf{s} = \mathbf{n}_T \times \mathbf{n}_B\, ds$. Therefore

$$i\lambda \oint_C d\mathbf{s} . \langle \mathbf{Q} \rangle = \oint_C \phi_T \mathbf{n}_T . \mathbf{n}_T \times \mathbf{n}_B\, ds + \oint_C \phi_B \mathbf{n}_B . \mathbf{n}_T \times \mathbf{n}_B\, ds$$

or

$$\oint_C d\mathbf{s} . \langle \mathbf{Q} \rangle = 0, \tag{4.19}$$

for $\lambda \neq 0$. *The mean circulation about the geostrophic contour C is zero for all inertial modes.* Only the geostrophic mode possesses any mean circulation and this property allows us to complete the synthesis of an arbitrary initial state of motion. Note that since h is constant on C, the actual average, $h^{-1}\langle \mathbf{Q} \rangle$ may be used in (4.19).

If the container is one for which no closed, constant-height contours exist, then there is, of course, no simple geostrophic mode. In this case, the theorem is not true (the proof fails) and the inertial modes can possess circulation. However, there is the possibility that an interior geostrophic motion can exist in such a configuration if non-linear inertial boundary layers arise to close the flow. The Gulf Stream in oceanic circulation exemplifies motion of this type.

Complete expansion theorem

If $\mathbf{U}(\mathbf{r})$ now represents a possible arbitrary velocity distribution inside a container, then it may be represented as a superposition of all the possible modes inertial and geostrophic

$$\mathbf{U}(\mathbf{r}) = \mathbf{q}_0 + \Sigma A_m \mathbf{Q}_m. \tag{4.20}$$

The mean circulation theorem is used first to determine $\mathbf{q}_0$ as follows: Integrate over the depth to obtain

$$\int_{-g}^{f} \mathbf{U}(\mathbf{r})\, dz = h\mathbf{q}_0 + \Sigma A_m \langle \mathbf{Q} \rangle$$

and compute the mean circulation about C,

$$\oint_C \langle \mathbf{U} \rangle . d\mathbf{s} = h \oint \mathbf{q}_0 . d\mathbf{s}. \tag{4.21}$$

From (3.8), it follows that

$$\partial\phi_0/\partial h = -\frac{2}{J} \oint_C \langle \mathbf{U} \rangle . d\mathbf{s}, \tag{4.22}$$

with
$$J = h \oint_C ds\{1+(\nabla f)^2\}^{\frac{1}{2}}\{1+(\nabla g)^2\}^{\frac{1}{2}} \tag{3.24}$$

and
$$\mathbf{q}_0 = -\tfrac{1}{2}(\partial\phi_0/\partial h)\{1+(\nabla f)^2\}^{\frac{1}{2}}\{1+(\nabla g)^2\}^{\frac{1}{2}}\, \mathbf{n}_T \times \mathbf{n}_B. \tag{3.7}$$

Having determined the geostrophic mode explicitly, the remaining Fourier coefficients are obtained using the orthogonality relationship (4.9)

$$A_m = \int (\mathbf{U} - \mathbf{q}_0) . \mathbf{Q}_m^+ \, dV \Big/ \int \mathbf{Q}_m^+ . \mathbf{Q}_m \, dV = \int \mathbf{U} . \mathbf{Q}_m^+ \, dV \Big/ \int \mathbf{Q}_m . \mathbf{Q}_m^+ \, dV \tag{4.23}$$

and this completes the modal synthesis of an arbitrary distribution. The full solution of the initial-value problem is displayed in the next section. The remainder of this section is devoted to the calculation of the viscous decay factor s_{m1} or rather a discussion of the computation. The solution of problems B_3 and B_4 along the lines laid out in the introduction and illustrated in I has been obtained by M. D. Kudlick, of the M.I.T. Mathematics Department, and it appears as a part of his doctoral thesis. Essentially, solutions of problems B_3 and B_4 must be found which are uniformly valid in time through spin-up. As we have remarked already, the factor s_{m1} can be derived by the classical procedure, provided all mathematical difficulties met *en route* are dismissed or ignored. Consider then problem B_4 with

$$\mathbf{q}_{m1} = \mathbf{Q}_{m1} e^{s_m t}. \tag{4.24}$$

If R is made infinite, the basic equations become

$$i\lambda_m \mathbf{Q}_{m1} + 2\mathbf{k} \times \mathbf{Q}_{m1} = -\nabla\Phi_{m1} - s_{m1}\mathbf{Q}_m, \quad \nabla . \mathbf{Q}_{m1} = 0, \tag{4.25}$$

with $\mathbf{n} . \mathbf{Q}_{m1}$ a known function on the boundary; say

$$\mathbf{n} . \mathbf{Q}_{m1} = F_m. \tag{4.26}$$

Of course, F_m is found by solving problem B_3 and represents the flux into the boundary layer that must arise when the primary internal tangential velocity is reduced to zero at the wall. If (4.25) is multiplied by $Q_m^+ .$ and the conjugate of (4.7) multiplied by Q_{m1} (in the manner used to establish the orthogonality relationship), the two may be added and integrated over the volume to obtain

$$\int \mathbf{Q}_m^+ . \nabla \Phi_m \, dV + \int \mathbf{Q}_{m1} . \nabla \Phi_m^+ \, dV + s_{m1} \int |\mathbf{Q}_m|^2 \, dV = 0. \tag{4.27}$$

Further simplification leads to the result

$$s_m = -\int \Phi_m^+ F_m \, dS / \int \mathbf{Q}_m^+ . \mathbf{Q}_m \, dV. \tag{4.28}$$

Kudlick has determined, after further reduction, that

$$-\int F_m \Phi_m^+ \, dS = -\frac{1}{2\sqrt{2}} \int \frac{dS}{1-(\mathbf{n}.\mathbf{k})^2} \left\{ |\mathbf{n}.\mathbf{k} \times \mathbf{Q}_m - i\mathbf{k}.\mathbf{Q}_m|^2 \, |p_1|^{\frac{1}{2}} \left(1 + \frac{ip_1}{|p_1|}\right) + |\mathbf{n}.\mathbf{k} \times \mathbf{Q}_m + i\mathbf{k}.\mathbf{Q}_m|^2 \, |p_2|^{\frac{1}{2}} \left(1 + \frac{ip_2}{|p_2|}\right) \right\}, \tag{4.29}$$

with $$p_{1,2} = \lambda_m \pm 2\mathbf{k}.\mathbf{n}.$$

In particular, this proves that

$$\operatorname{Re} s_{m1} < 0,$$

as it must be. A special case of this result, with several numerical computations for spherical modes, is contained in I; Kudlick has also investigated other configurations.

The eigenfunctions corresponding to $|\lambda| = 2$ fit into the theory presented here with no special attention required. However $\mathbf{Q}$ cannot be determined in entirety from the pressure function alone for such an attempt introduces the factor $(1-\frac{1}{4}\lambda^2)$, in this case a zero quantity. As a matter of fact, if $\lambda = 2$, say, then it follows directly that

$$\nabla \Phi - (\mathbf{k}.\nabla\Phi)\mathbf{k} + i\mathbf{k} \times \nabla\Phi = 0, \tag{4.30}$$

everywhere inside the container. This is a much more restrictive requirement than equations (4.3), (4.4), with which (4.30) is completely consistent. The fact that Φ satisfies a lower-order equation when $\lambda = 2$ probably means that in this case there is no acceptable solution other than zero. Certainly, for axially symmetric containers this is indeed true, but the proof will not be given here.

The geostrophic mode may be viewed as the totality of all inertial modes corresponding to $\lambda = 0$. Each mode would then represent a non-zero flow only in a cylindrical shell of infinitesimal cross-section about a geostrophic contour C, with generators parallel to $\mathbf{k}$. The decay rate of such an infinitesimal mode obviously depends only on the local value of h. The sum (or integral) of these modes creates the general geostrophic flow whose decay rate is a function of position.

The eigenvalue spectrum for the inviscid interior problem is known from I to be denumerable but dense in the interval $|\lambda| \leqslant 2$. This has been shown to be only a manifestation of non-uniform limit processes; the spectrum really consists of isolated singularities. Actually, the entire concept upon which the inviscid analysis is based, that each mode is separable into an inviscid interior with a boundary-layer correction, breaks down when the effective modal wavelength is of the order of the boundary-layer thickness. If μ is a typical wave-number, the precise eigenvalue location involves the factor $iR^{-1}\mu^2$. Clearly then the error made in locating λ by setting $R = \infty$ is order one for values $\mu = O(R^{\frac{1}{2}})$. The conclusion that the spectrum is dense results from the failure of the asymptotic method to locate the eigenvalues properly. The approximation yields only the projection of the correct eigenvalue position on to the real λ axis; the projected positions are dense.

5. The initial-value problem

The interior solution of the initial-value problem, uniformly valid in time through the spin-up time to order $(R^{-\frac{1}{2}})$, may now be displayed in full. The general time-dependent solution for the velocity vector is

$$\mathbf{q}(\mathbf{r},t) = \mathbf{q}_0(x,y,R^{-\frac{1}{2}}t) + \Sigma A_m \mathbf{Q}_m(\mathbf{r}) \exp[(i\lambda_m + s_{m1}R^{-\frac{1}{2}})t], \tag{5.1}$$

where
$$\mathbf{q}_0 = -\tfrac{1}{2}\{1+(\nabla f)^2\}^{\frac{1}{2}}\{1+(\nabla g)^2\}^{\frac{1}{2}}(\partial\phi_0(h,R^{-\frac{1}{2}}t)/\partial h)\,\mathbf{n}_T\times\mathbf{n}_B, \tag{3.7}$$

$$\frac{\partial\phi_0(h,R^{-\frac{1}{2}}t)}{\partial h} = \frac{\partial\phi_0(h,0)}{\partial h}\exp\left(-\frac{I}{J}R^{-\frac{1}{2}}t\right); \tag{3.23}$$

and
$$\left.\begin{aligned} I &= \oint_C ds\{1+(\nabla f)^2\}^{\frac{1}{2}}\{1+(\nabla g)^2\}^{\frac{1}{2}}[\{1+(\nabla f)^2\}^{\frac{1}{4}}+\{1+(\nabla g)^2\}^{\frac{1}{4}}],\\ J &= h\oint_C ds\{1+(\nabla f)^2\}^{\frac{1}{2}}\{1+(\nabla g)^2\}^{\frac{1}{2}}.\end{aligned}\right\} \tag{3.24}$$

If $\mathbf{q}(\mathbf{r},0) = \mathbf{q}_*(\mathbf{r})$ then

$$\mathbf{q}_*(\mathbf{r}) = \mathbf{q}_0(x,y,0) + \Sigma A_m \mathbf{Q}_m(\mathbf{r}); \tag{5.2}$$

the mean circulation theorem and orthogonality relationships are used to determine the unknown quantities. Therefore

$$\frac{\partial\phi_0(h,0)}{\partial h} = -\frac{2}{J}\oint_C \langle\mathbf{q}_*\rangle . ds \tag{5.3}$$

and
$$A_m = \int \mathbf{q}_*(\mathbf{r}).\mathbf{Q}_m^+ dV / \int \mathbf{Q}_m^+ . \mathbf{Q}_m dV. \tag{5.4}$$

The pressure is then

$$p(\mathbf{r},t) = \phi_0(x,y,R^{-\frac{1}{2}}t) + \Sigma B_m \Phi_m(\mathbf{r}) \exp[(i\lambda_m + s_{m1}R^{-\frac{1}{2}})t]. \tag{5.5}$$

It must be recognized that, in any specific application (as in I), the pressure is determined first and is the primary solution function. The velocity is then calculated to complete the solution, and the coefficients A_m and B_m are related through equations (4.1) or (4.2).

In a container having closed contours of constant height h, the geostrophic mode is excited by any initial velocity distribution possessing mean circulation. That part of the initial flow with no mean circulation stimulates inertial oscillations, all of which decay in the time $R^{\frac{1}{2}}\Omega^{-1}$, as does the geostrophic flow. If the configuration does not allow geostrophy to occur, the inertial modes can possess mean circulation and seem capable, in themselves, of representing any initial distribution. The possibility of a geostrophic interior joined to a non-linear inertial boundary layer, as in oceanic circulation, cannot be overlooked or ignored.

Viscosity, by means of the Ekman boundary layer, always acts to establish a state of rigid rotation in the time $R^{\frac{1}{2}}\Omega^{-1}$. This rotational viscous layer necessitates a small normal mass flux and thereby generates an interior circulation that convects angular momentum and stretches vortex lines. Thus viscosity strongly affects the interior régime, not by viscous diffusion, but rather by the processes of momentum transport and vortex line stretching. This accounts for the comparatively short characteristic time scale to destroy deviations from rigid rotation. This description has been given before for special configurations; the present work shows that it is generally true for contained rotating fluid motions.

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REFERENCES

Greenspan, H. P. 1964 *J. Fluid Mech.* **20**, 673.

Greenspan, H. P. & Howard, L. N. 1963 *J. Fluid Mech.* **17**, 385.

Greenspan, H. P. & Weinbaum, S. 1965 *J. Math. & Phys.* (To be published.)